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Liquidity without Money: A General Equilibrium Model of Market Microstructure*

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We consider a model in which the market period is divided into T rounds of trading, with the arrival of consumers determined exogenously. A monopolistic market maker sets the bid-price and the ask-price in each round, accepting all trades at the stated prices. Optimal prices remain constant when the aggregate supply and demand are known to the market maker, even if supplies and demands within individual rounds are not known. Examples illustrate the endogenous determination of inventory holding costs. The viability of a market-maker system is compared to an auction market with and without a "sophisticated" trader. *Journal of Economic Literature* Classification Numbers: 021, 022, 026.

1. INTRODUCTION

With complete markets, a competitive equilibrium results in a Pareto optimal allocation, and the liquidity of markets is an irrelevant concept. This paper begins with the view that markets are incomplete and provides a general equilibrium analysis of market liquidity and the market making function. Liquidity is provided in a model without money. The paper

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contributes to the market microstructure literature in the following ways: (1) It provides an alternative, endogenous explanation for inventory holding costs with results similar to the existing literature. (2) It shows that the approach applies equally well to markets where there is certainty about asset payoffs. (3) It provides a model where all the traders are rational, so that welfare analysis is possible. (4) It compares alternative market microstructures.

Consider a group of traders, some of whom want to give up current consumption in exchange for future consumption and some of whom want to give up future consumption in exchange for current consumption. Suppose that the market period consists of T subperiods, or trading rounds, and that the number of consumers that arrive during a particular round is exogenous and stochastic.

We consider the following transactions technology. An auctioneer can be costlessly engaged to select the market clearing price through a tâtonnement process. Maintaining a continuous market presence and learning the stochastic process for supply and demand are costly. Furthermore, it is prohibitively expensive to collect and process demand schedules at the end of the trading period (which would complete the market). An auctioneer who is paid to clear markets and not to study them does not know the true Walrasian price until after the fact, and if he equates demand and supply in each round, the price will fluctuate as the number of borrowers versus savers fluctuates. A need emerges for a market maker to bridge temporary imbalances between the number of borrowers and the number of savers, providing liquidity to the market.

Completing the description of the transactions technology involves specifying the set of feasible market institutions, which is a formidable task. Instead, we focus on the case of all trade passing through a market maker, who sets a bid- and ask-price. The market maker can change prices from round to round, but he must accept all customers at the posted prices within a round.

In Section 2, we solve the static model with one trading round. Section 3 explores a multi-round model. When there is no uncertainty about the total demand and supply (although there may be uncertainty about demand or supply within a particular round), a "law of two prices" result is proved. That is, the market maker optimally chooses the bid-price and ask-price to be constant across time. An example is provided where there is aggregate uncertainty as well as within-round uncertainty. Prices fluctuate as the inventory holding cost endogenously fluctuates. When more borrowers (sellers of future consumption) than expected arrive in the first round, the market maker lowers his bid-price and his ask-price (equivalently, he raises the interest rate he pays and the interest rate he charges).

In Section 4, we consider an example with no uncertainty about total supply and demand. The example is used to consider the viability of a market maker over an auctioneer who clears the market round-by-round. For some parameter values, the benefit to consumers of price smoothing outweighs the cost of allowing the market maker his monopoly profit. Section 5 explores the robustness of this result by considering an auction market with a "sophisticated" trader who participates in each round's auction after paying the fixed cost of learning the stochastic process of supply and demand. These preliminary results suggest that the market maker will tend to predominate when the imbalances between supply and demand within a round are great, when supply and demand aggregated over all rounds is predictable, and when the cost of becoming informed about the arrival process is moderate. Section 6 presents some concluding remarks.

There is a related literature on the behavior of market makers in securities markets, starting with Garman (1976) and continuing with Stoll (1978), and Ho and Stoll (1981). These models illustrate the trade-offs faced by the market maker. In Ho and Stoll (1981), holding costs arise from the return risk faced by risk averse market makers. They assume that the time horizon is small enough so that inventory holdings are never completely depleted, arguing that the market maker can always adjust the price to avoid such a problem. They find that prices move up and down as inventories are depleted or replenished, with the bid–ask spread remaining constant. This paper focuses on the opposite extreme in which a market maker faces no return risk but a significant risk of being forced to honor his promises by adversely changing the price. That is, holding costs are completely endogenous and arise when the (risk neutral) maker is caught having to close out a position. Many of our results reinforce those of Ho and Stoll.

Glosten and Milgrom (1985) consider a market where some insiders have superior information to which noise traders and market makers are not privy. Market makers earn zero expected profits while the noise traders pay for the profits of the insiders. Kyle (1985) considers a similar model with a single insider who takes into account the effects of his actions on future prices. In both papers, the noise traders are not explicitly optimizing, which makes welfare analysis difficult. Another difference between these papers and the present papers is that here the asset pays one unit of consumption in all states, so that adverse selection is not a problem.

Starr (1987) sketches a model that captures the need for financial intermediaries or secondary markets to provide liquidity. Individual wealth holders desire assets that can be immediately transformed into consumption. Although the arrival of an individual to liquidate his wealth is random, aggregate demand for liquidation is regular and known. Financial intermediaries offer wealth holders a superior alternative to low-yield, short-maturity investments by offering them liquid claims on high-yield, long-lived physical assets. While Starr (1987) focuses on imbalances between the length of time a wealth holder wants to hold an asset and the gestation period of physical capital, the present paper focuses on imbalances in the arrival of traders that arise within the market period.

2. THE STATIC MODEL

There are two market periods and one consumption good per period. In this section, period 1 consists of a single round of trading. First-period consumption, the numeraire, is traded for a security or bond which pays one unit of second-period consumption in all states of nature.

There are three types of individuals in this economy, suppliers of bonds (borrowers), demanders of bonds (savers), and a single market maker. The market maker sets the price at which suppliers can sell bonds, p^{bid}, and the price at which demanders can buy bonds, p^{ask}. It is assumed that all trade is done through the market maker and that the market maker must accept all customers at its stated price. That is, in all states of nature, the market maker must have purchased enough first-period consumption to satisfy his obligations to the suppliers of bonds. To emphasize the liquidity-providing role of the market maker, we suppose he only cares about second-period consumption. Thus, a market maker with no endowment of the numeraire commodity would have no incentive to trade if he were an ordinary consumer. We do not restrict second-period consumption to be nonnegative; one interpretation is that the market maker has a large enough endowment of second-period consumption to guarantee that a second-period constraint would not be binding. The market maker is therefore concerned about insolvency but not bankruptcy.

We assume that there is a technology for costlessly storing the commodity. Such an assumption is not essential, but it allows the model to resemble a partial equilibrium formulation focusing on the market between bonds and money. Stored consumption represents the money that the market maker holds but does not rely upon (precautionary demand); this inventory can be held from one period to the next without spoilage.

Demanders and suppliers face no uncertainty, so they maximize their utility subject to a budget constraint. We impose conditions on demand (as a function of p^{ask}) and supply (as a function of p^{bid}). However, a broad class of utility functions and endowments satisfies these conditions, including those of the examples. We assume that demand and supply are twice continuously differentiable and that we have

$$D' < 0,\ D'' \geq 0,\ D'' + 2D' < 0,\ D(p^{\text{ask}} = 1) = 0,$$
$$S' > 0,\ S'' \leq 0,\ S'' + 2S' > 0, \text{ and } S(p^{\text{bid}} = 0) = 0.$$

The only uncertainty in the model is that the market maker does not know the number of buyers and sellers who will arrive during the trading round. Supply is given by

$$F(p^{\text{bid}}, \theta) = \begin{cases} \theta_1 S(p^{\text{bid}}), & \text{with probability } q_1 \\ \theta_2 S(p^{\text{bid}}), & \text{with probability } q_2 \\ \vdots & \\ \theta_m S(p^{\text{bid}}), & \text{with probability } q_m, \end{cases} \tag{1}$$

and demand is given by

$$G(p^{\text{ask}}, \phi) = \begin{cases} \phi_1 D(p^{\text{ask}}), & \text{with probability } r_1 \\ \phi_2 D(p^{\text{ask}}), & \text{with probability } r_2 \\ \vdots & \\ \phi_m D(p^{\text{ask}}), & \text{with probability } r_m, \end{cases} \tag{2}$$

Let θ be a random variable representing the number of suppliers, where the possible realizations for θ are $\theta_1, \theta_2, \ldots, \theta_{\text{m}}$; let ϕ represent the number of demanders, with possible realizations $\phi_1, \phi_2, \ldots, \phi_{\text{n}}$. For this section, we assume that supply and demand shocks are independent, that $\Sigma_{i=1}^{m} q_i = 1$ and that $\Sigma_{j=1}^{n} r_j = 1$. Without loss of generality, let $\theta_1 < \cdots < \theta_{\text{m}}$ and $\phi_1 < \phi_2 < \cdots < \phi_n$. Let I denote the market maker's inventory of first-period consumption. The risk-neutral market maker's optimization problem is

$$\max_{p^{\text{bid}}, p^{\text{ask}}} \sum_{i=1}^{m} q_i(1 - p^{\text{bid}})\, S(p^{\text{bid}})\, \theta_i - \sum_{j=1}^{n} r_j(1 - p^{\text{ask}})\, D(p^{\text{ask}})\, \phi_j \tag{3}$$

$$\text{s.t. } p^{\text{bid}}\, \theta_m S(p^{\text{bid}}) \leq p^{\text{ask}}\, \theta_1 D(p^{\text{ask}}) + I, \qquad 0 \leq p^{\text{bid}} \leq 1,\ 0 \leq p^{\text{ask}} \leq 1. \tag{4}$$

The first term in expression (3) represents the expected revenue received from issuing loans, and the second term represents the expected cost incurred from obtaining first-period consumption. The constraint (4) requires that even when there are many borrowers and few savers, the market maker must have enough first-period consumption to satisfy all the borrowers.

If the market maker starts with enough inventory, we will have a corner solution in which $p^{\text{ask}} = 1$ (the market maker does not need demanders) and p^{bid} is the price that maximizes $(1 - p^{\text{bid}})\, S(p^{\text{bid}})$. Letting $\bar{p}^{\text{bid}}$ denote the unconstrained expected profit maximizing price, then the maximization problem (3)–(4) has a corner solution whenever $I > \bar{p}^{\text{bid}}\, \theta_m S(\bar{p}^{\text{bid}})$ holds. This solution is uninteresting because there is no connection between the supply side and the demand side of the market.

Whenever $I < \bar{p}^{\text{bid}}\, \theta_m S(\bar{p}^{\text{bid}})$, we have an interior solution in which $p^{\text{bid}} < \bar{p}^{\text{bid}}$ and $p^{\text{ask}} < 1$. The necessary and sufficient first-order conditions for an optimum are (4) holding with equality and

$$\lambda = \left[\frac{(1 - p^{\text{bid}})\, S'(p^{\text{bid}}) - S(p^{\text{bid}})}{p^{\text{bid}}\, S'(p^{\text{bid}}) + S(p^{\text{bid}})}\right] \frac{\sum_{i=1}^{m} q_i\theta_i}{\theta_m} \tag{5}$$

$$\lambda = \left[\frac{(1 - p^{\text{ask}})\, D'(p^{\text{ask}}) - D(p^{\text{ask}})}{p^{\text{ask}}\, D'(p^{\text{ask}}) + D(p^{\text{ask}})}\right] \frac{\sum_{j=1}^{n} r_j\phi_j}{\phi_1}, \tag{6}$$

where λ is the Lagrangean multiplier on constraint (4).

The following proposition is the main result of this section.

PROPOSITION 2.1. *Given our assumptions about supply and demand, at any interior equilibrium we have*

(a) $\partial p^{\text{ask}}/\partial I > 0$ *and* $\partial p^{\text{bid}}/\partial I > 0$.

(b) *A proportional increase in* $\theta_1, \theta_2, \ldots, \theta_m$ *results in lower prices* (p^{bid} *and* p^{ask}) *and higher interest rates.*

(c) *A proportional increase in* $\phi_1, \phi_2, \ldots, \phi_n$ *results in higher prices* (p^{bid} *and* p^{ask}) *and lower interest rates.*

The proof of Proposition 2.1 is given in the Appendix. The comparative statics results of changing the market maker's inventory, the number of buyers, or the number of sellers affect p^{ask} and p^{bid} in predictable ways. Note that the market maker's action on the supply side of the market depends on the parameters of the demand side and vice versa. Parts (b) and (c) of Proposition 2.1 say that the more borrowers, the higher the interest rate paid to lenders and the more lenders, the lower the interest rate charged to borrowers. This intertwining of supply and demand considerations is what establishes the market maker as an intermediary rather than a provider of two unrelated services.[1]

Constraint (4) forces the market maker to acquire enough first-period consumption to satisfy all borrowers in all states of nature. If the maximum number of borrowers or the minimum number of lenders is high, and these probabilities are low, then the outcome is inefficient. Meeting the supply leads to a lower shadow price on inventories, which implies a bigger spread (see Eqs. (5) and (6)). Allowing for the possibility of stockouts, in which the net demand for first-period consumption exceeds the market maker's inventory, could lead to a Pareto improvement (in terms of expected utility). There is a trade-off between having a bigger spread and decreasing the probability of being rationed (as stockouts occur in fewer states). Analyzing the possibility of stockouts is worthwhile but would constitute a project in and of itself, and is not attempted here.

In the multi-round model, the market maker can adjust prices before his inventory is exhausted. He can temporarily maintain a short position, adjusting prices if necessary to acquire all the first-period consumption he has promised to sell. These possibilities make it less crucial to allow stockouts in the multi-round setting.

3. THE MULTI-ROUND MODEL

The multi-round model permits the market maker to change prices as customers arrive within the market period. Consider the following time line:

Time line

Period 1 Period 2

× ——— × ——— × ——— · · · ——— × ———|——— × ———

Round 1 Round 2 Round 3 Round T

First-period consumption changes hands

Bonds pay 1 unit of second-period consumption

During each round in period 1, the same commodity is being traded. Since delivery occurs after round T, the market maker can hold a short position

[1] From Proposition 2.1, we can show that the effect of I on overall economic welfare is ambiguous, with the results depending on the social welfare function chosen. Since increases in I increases both p^{bid} and p^{ask}, demanders of bonds are made worse off and suppliers of bonds are made better off. At any interior solution, higher inventories allow the market maker to raise p^{ask} while still maintaining constraint (4), leading to higher profits. (In fact, he does even better by raising p^{ask} and p^{bid}.)

as long as he meets his delivery obligations in all states. The only features that delineate one round from another are the customers who arrive within that round and the corresponding information received by the market maker. The arrival of consumers in each round is stochastic and is independent of the market maker's actions.

Let T denote the number of rounds, with t indexing a particular round. Supply and demand are given by

$$F_t(p_t^{\text{bid}}, \omega) = \theta^t S(p_t^{\text{bid}})$$

$$G_t(p_t^{\text{ask}}, \omega) = \phi^t D(p_t^{\text{ask}}),$$

where $\theta^t \in \{\theta_1, \theta_2, \ldots, \theta_m\}$ and $\phi^t \in \{\phi_1, \phi_2, \ldots, \phi_n\}$. The state of nature, ω, is composed of all the shocks, so we have $\omega = (\theta^1, \theta^2, \ldots, \theta^T, \phi^1, \phi^2, \ldots, \phi^T)$. The joint distribution of shocks is given by the probability function $f(\omega)$, and the set of states is denoted by Ω.

The market maker's problem is to choose prices in each period to maximize expected utility subject to the solvency constraint. These prices can be functions of previous demands and supplies. More formally,

$$\max \Sigma_{\omega\in\Omega} f(\omega)[\Sigma_{t=1}^{T} (1 - p_t^{\text{bid}})\, \theta^t S(p_t^{\text{bid}}) - \Sigma_{t=1}^{T} (1 - p_t^{\text{ask}})\, \phi^t D(p_t^{\text{ask}})] \tag{7}$$

subject to

$$0 \le p_t^{\text{bid}} \le 1,\ 0 \le p_t^{\text{ask}} \le 1;$$

$$\Sigma_{t=1}^{T} p_t^{\text{bid}}\, \theta^t S(p_t^{\text{bid}}) \le I + \Sigma_{t=1}^{T} p_t^{\text{ask}}\, \phi^t D(p_t^{\text{ask}}), \qquad \text{for all } \omega \in \Omega; \tag{8}$$

p_t^{bid} and p_t^{ask} are measurable functions of ω, reflecting knowledge of the history of shocks through period $t - 1$. (9)

In general, the market maker will vary prices, raising interest rates when inventory is low relative to expected future supply and demand, and lowering interest rates when inventory is in excess. An example is presented later in this section. However, there is a special case in which the market maker will maintain constant prices throughout. This is the case in which the total number of customers, $\Sigma_{t=1}^{T} \theta^t$ and $\Sigma_{t=1}^{T} \phi^t$, is known at the outset. Prices are constant when there is no aggregate uncertainty, no matter how uncertain the market maker is about any particular round and no matter how likely it is that supply or demand is high in one round and low in another.[2]

PROPOSITION 3.1 (The Law of Two Prices). *Suppose* $\sum_{t=1}^{T} \theta^t$ *and* $\sum_{t=1}^{T} \phi^t$ *are known. Then prices are constant:* $p_t^{\text{bid}} = \overset{*}{p}{}^{\text{bid}}$ *and* $p_t^{\text{ask}} = \overset{*}{p}{}^{\text{ask}}$ *for* $t = 1, 2, \ldots, T$ and for all $\omega \in \Omega$. *Furthermore,* $\overset{*}{p}{}^{\text{bid}}$ *and* $\overset{*}{p}{}^{\text{ask}}$ *are the equilibrium prices of the one-round model with (known) supply* $(\sum_{t=1}^{T} \theta^t)S(p^{\text{bid}})$ *and demand* $(\sum_{t=1}^{T} \phi^t)D(p^{\text{ask}})$.

The proof of Proposition 3.1 is in the Appendix. The intuition is as follows: even if the market maker could select different prices for each consumer, identical consumers will be treated identically. Thus, the optimal solution under perfect certainty and the ability to price discriminate is to select a single bid-price and a single ask-price. Since these prices only depend on $\sum_{t=1}^{T} \theta^t$ and $\sum_{t=1}^{T} \phi^t$, the first-best solution is feasible in the more tightly constrained problem (7)–(9), so it must be optimal. If many demanders arrive early, the effect of higher inventories exactly offsets the effect of more suppliers arriving in the future.

The case of no aggregate uncertainty represents the extreme in which the market maker provides perfect liquidity to the consumers. By being willing to take the other side of the market when demand and supply imbalances emerge, the market maker provides both "depth" and "continuity." With a Walrasian auctioneer, a big trader will not want to place a big order all at once for fear of adversely affecting the price. Here, a big trader is not inconvenienced by being forced to trade during just one round because of the depth provided. Similarly, a trader need not fear price jumps if an immediate need to trade arises. Because prices are constant, we have perfect continuity.

Allowing short sales is reasonable for many applications, but we must answer why demand and supply schedules cannot be collected by a Walrasian auctioneer.[3] Presumably, information-processing constraints prevent the market maker (or a Walrasian auctioneer) from collecting demand and supply schedules, with prices chosen at the end of the last round. Requiring nonnegative inventories would allow us to interpret the need to transact as instantaneous, so that aggregation across rounds is impossible, independent of information-processing capabilities. A nonne-

[2] This result depends on the multiplicative nature of the shocks. That is, demand and supply within a period are random but demand and supply elasticities are not. Otherwise the market maker can, in effect, price discriminate.

[3] One could argue that certain knowledge of the transaction is needed immediately, while the actual consumption occurs at the end of the market period. For example, this model could be a reduced form of another model in which consumers take other actions after knowing their trade.

gative inventory constraint would not change the role of the market maker as a provider of liquidity, although Proposition 3.1 might not hold if the constraint becomes binding before the last round.

PROPOSITION 3.2. *Suppose the market maker is constrained to hold nonnegative inventories round-by-round. That is, constraint* (8) *is replaced with*

$$\sum_{t=1}^{T-\tau} p_t^{\text{bid}}\theta^t S(p_t^{\text{bid}}) \leq I + \sum_{t=1}^{T-\tau} p_t^{\text{ask}}\phi^t D(p_t^{\text{ask}})$$
$$\text{for } \tau = 0, 1, \ldots, t-1 \text{ and for all } \omega \in \Omega. \quad (10)$$

Suppose also that we have, for $\tau = 1, 2, \ldots, T-1$ *and for all* $\omega \in \Omega$,

$$\sum_{t=1}^{T-\tau} \phi^t / \sum_{t=1}^{T-\tau} \theta^t \geq \sum_{t=1}^{T} \phi^t / \sum_{t=1}^{T} \theta^t. \quad (11)$$

Then, when $\sum_{t=1}^{T} \theta^t$ *and* $\sum_{t=1}^{T} \phi^t$ *are known, prices are constant and the solution is exactly the same as the solution to problem* (7)–(9).

The proof of Proposition 3.2 is in the Appendix. Condition (11) requires that for all histories, the cumulative number of demanders divided by the cumulative number of suppliers is greater than or equal to the known ratio of total demanders to total suppliers.

If Condition (11) is violated, then bid- and ask-prices will typically not be constant. For example, suppose there is no uncertainty, with the sequence of suppliers known to be (5, 10, 5, 5, . . . , 5) and the sequence of demanders known to be (5, 5, . . . , 5, 10), and suppose I = 0. The optimal solution involves constraint (10) binding for $t = 2$ and for $t = T$. Since constraint (10) is binding for $t = 2$, no inventories will be carried into round 3. Since all inventories are sold at the end of round 2, the optimal prices in rounds 1 and 2 coincide with the solution to the analogous two-round model where (5, 10) is the vector of suppliers and (5, 5) is the vector of demanders. The problem is effectively split into a two-round problem followed by a $(T-2)$-round problem. For each of these subproblems, condition (11) is satisfied. Therefore, prices will be constant in rounds 1 and 2, and they will be constant for rounds 3 to T. Because the first two rounds have a higher ratio of supply to demand than the last $T-2$ rounds, the bid-price and the ask-price are lower in rounds 1 and 2 than in rounds 3 to T.

With uncertainty within a round (while maintaining the assumption of no aggregate uncertainty), the situation is more complicated, but some of the intuition remains from the case of perfect foresight. For some histories, either condition (11) will be satisfied or inventories will be large enough so that prices become constant from that point forward. A set of "subgames" is isolated, where constraint (10) is never binding (until the last round). For histories where constraint (10) could subsequently become binding, two effects are present which cause prices to fluctuate. A high realization of net supply brings the market maker's inventories closer to zero, which increases the likelihood of hitting zero before demand recovers, other things being equal. This effect increases the shadow-price of inventories and induces the market maker to lower the bid- and ask-price. At the same time, the current realizations can provide information about the timing of future arrivals (known in the aggregate). This effect can go in either direction. For example, suppose the highest number of suppliers and lowest number of demanders arrive in round t, and that this event guarantees that all remaining demanders arrive in round $t + 1$. In effect, the precautionary demand for inventories is eliminated, so the shadow-price cannot go up.

The following example illustrates the typical pattern of price movements with aggregate uncertainty. These price movements are in accord with the existing literature on market microstructure. However, as the previous discussion suggests, the qualitative results could differ when shocks are not temporally independent.

EXAMPLE 1. Preferences are linear-quadratic, described by the utility function

$$u(c_1, c_2) = c_1 + c_2 - (c_2)^2.$$

There are two types of consumers, distinguished only by their endowments. Demanders of second-period consumption have an endowment given by

$$w(\text{demanders}) = (w_1, w_2) = (.5, 0).$$

Suppliers of second-period consumption have an endowment given by

$$w(\text{suppliers}) = (w_1, w_2) = (.5, .5).$$

Consumers maximize utility subject to their budget constraints. Consumers with strictly positive endowments can transact at either p^{ask} or p^{bid}, depending on the direction of their trade, so the budget constraint can have a kink. Given the endowment vector (w_1, w_2), the budget constraint is

$$\begin{aligned}
&c_1 - w_1 + p^{\text{ask}}(c_2 - w_2) \leq 0 \qquad \text{if } c_2 - w_2 \geq 0,\\
&c_1 - w_1 + p^{\text{bid}}(c_2 - w_2) \leq 0 \qquad \text{if } c_2 - w_2 \leq 0,\\
&c_1 \geq 0,\ c_2 \geq 0.
\end{aligned}$$

It is straightforward to show that those we call demanders will choose to buy (at p^{ask}) and those we call suppliers will choose to sell (at p^{bid}) whenever prices are strictly between zero and one.

Solving for the demand function of a representative lender, we have

$$D(p^{\text{ask}}) = c_2 - w_2 = .5 - .5\, p^{\text{ask}}, \tag{12}$$

which clearly satisfies all of our assumptions. The supply function of a representative borrower is given by

$$S(p^{\text{bid}}) = w_2 - c_2 = .5\, p^{\text{bid}}, \tag{13}$$

which also satisfies our assumptions.

There are two types of shocks, α and β. If α occurs, there are $2 + 2\gamma$ borrowers and $2 - 2\gamma$ savers, where γ is a known parameter. That is, we have for all t,

$$\begin{aligned} G_t(p_t^{\text{ask}}, \alpha) &= (1 - \gamma)(1 - p_t^{\text{ask}}) \\ F_t(p_t^{\text{bid}}, \alpha) &= (1 + \gamma)p_t^{\text{bid}}. \end{aligned} \tag{14}$$

If β occurs, there are $2 - 2\gamma$ borrowers and $2 + 2\gamma$ savers, so we have for all t,

$$\begin{aligned} G_t(p_t^{\text{ask}}, \beta) &= (1 + \gamma)(1 - p_t^{\text{ask}}) \\ F_t(p_t^{\text{bid}}, \beta) &= (1 - \gamma)p_t^{\text{bid}}. \end{aligned} \tag{15}$$

There are two rounds and shocks are independent and identically distributed. The probability of each shock is .5. Therefore, overall supply and demand are random. To provide the most direct comparison between the market maker and the auctioneer, we assume $I = 0$, so the market maker has no motive to trade except as an intermediary. Let p_1^{ask} denote the ask-price in round 1, p_α^{ask} denote the ask-price in round 2 if α occurs in round 1, and p_β^{ask} denote the ask-price in round 2 if β occurs in round 1, with the corresponding notation for bid-prices. The market maker chooses p_1^{bid}, p_1^{ask}, p_α^{bid}, p_α^{ask}, p_β^{bid}, and p_β^{ask} to solve

$$\begin{aligned} \max(1 - p_1^{\text{bid}})p_1^{\text{bid}} - (1 - p_1^{\text{ask}})^2 + (1 - p_\alpha^{\text{bid}})p_\alpha^{\text{bid}}/2 \\ - (1 - p_\alpha^{\text{ask}})^2/2 + (1 - p_\beta^{\text{bid}})p_\beta^{\text{bid}}/2 - (1 - p_\beta^{\text{ask}})^2/2 \end{aligned} \tag{16}$$

subject to

$$(p_\alpha^{\text{bid}})^2(1+\gamma) - p_\alpha^{\text{ask}}(1-p_\alpha^{\text{ask}})(1-\gamma)$$
$$= p_1^{\text{ask}}(1-p_1^{\text{ask}})(1-\gamma) - (p_1^{\text{bid}})^2(1+\gamma); \quad (17)$$

$$(p_\beta^{\text{bid}})^2(1+\gamma) - p_\beta^{\text{ask}}(1-p_\beta^{\text{ask}})(1-\gamma)$$
$$= p_1^{\text{ask}}(1-p_1^{\text{ask}})(1+\gamma) - (p_1^{\text{bid}})^2(1-\gamma); \quad (18)$$

and prices must be between zero and one.

We consider the case of $\gamma = .5$. We have

$$\begin{aligned} p_1^{\text{bid}} &= .2545,\ p_\alpha^{\text{bid}} = .1782,\ p_\beta^{\text{bid}} = .3644 \\ p_1^{\text{ask}} &= .8362,\ p_\alpha^{\text{ask}} = .8121,\ p_\beta^{\text{ask}} = .9448, \end{aligned} \quad (19)$$

and the market maker's expected profits are .3327.

Equations (19) illustrate the typical pattern of prices with aggregate uncertainty. When α occurs in round 1, supply is high and demand is low relative to average levels for round 1. When β occurs in round 1, demand is high and supply is low relative to average. Thus, the inventory carried into round 2 is more than the market maker would have chosen had he known α would occur in round 1, and it is less than he would have chosen had he known β would occur. In other words, when α occurs in round 1, inventories are plentiful and the market maker lowers both prices in round 2. When β occurs in round 1, the market maker finds himself strapped for first-period consumption and he raises both prices.

The driving force behind these price movements is the delivery constraint (8). Being caught after round 1 with too much inventory forces the market maker to sell at a lower price than he otherwise would have received. Being caught with too small an inventory (perhaps negative) forces the market maker to offer a higher price than he otherwise would have offered. This example suggests that in markets where liquidity is important and there is a time at which the market closes and commitments are honored, "backwardation" can become an important consideration.

The price movements of Example 1 are similar to those of Ho and Stoll (1981), who consider a continuous time model with demand and supply modeled as a Poisson jump process. The most significant difference between the two models is that holding costs are attributed to a risk averse market maker facing return risk in Ho and Stoll (1981). They also find that prices move up and down as inventories are depleted or replenished, with the bid–ask spread remaining constant. Here, the return of the asset is completely known. What generates the price swings and the need to service both sides of the market is the constraint that the market maker must acquire enough first-period consumption to honor his commitments. When this constraint comes into play, we find price movements similar to those of Ho and Stoll (1981) (although their constant spread result does not generalize to arbitrary demands and distributional assumptions). Thus, if they had taken into account the possibility of inventories hitting zero, their results would probably not have changed very much. Furthermore, Example 1 indicates that the desire of a market maker to maintain a balanced position could arise even when he is risk neutral.

4. COMPARING THE MARKET MAKER TO A WALRASIAN AUCTIONEER

In this section we compare the market maker system to a system of round-by-round market clearing by a Walrasian auctioneer. The auctioneer does not invest the effort to analyze his market; he mechanically clears the market without trading on his own account. Clearing the market via tâtonnement is costless, although collecting and processing demand schedules is prohibitively expensive. To justify truth-telling, think of each trader as being small relative to the market.

In Example 2, there is no aggregate uncertainty. Proposition 3.1 has implications for the desirability of a market maker over a system of round-by-round market clearing by a Walrasian auctioneer. When supply sometimes outpaces demand, but demand sometimes outpaces supply as well, the Walrasian auctioneer results in excessive price swings from round to round. Significant gains from trade remain unexploited. The market maker would smooth any temporary imbalances between supply and demand by offering trade at constant prices. For some parameter values the benefits of price smoothing exceed the costs of paying the market maker his monopoly profit.

EXAMPLE 2. This example has the same types of shocks and preferences as Example 1. There are two rounds and, unlike the previous example, there is no aggregate uncertainty.[4] In particular, it is known that α occurs once and β occurs once. To provide the most direct comparison between the market maker and the auctioneer, we assume $I = 0$, so the market maker has no motive to trade except as an intermediary.

[4] The number of rounds can be arbitrary, as long as half of them are α and half of them are β. Specifying two rounds allows a ready comparison of profits to those in Example 1.

It is straightforward to solve for the equilibrium prices, using Proposition 3.1 and Eqs. (4)–(6). The solution is, for $t = 1, 2$,

$$\begin{aligned} p_t^{\text{bid}} &= .35355 \\ p_t^{\text{ask}} &= .85355. \end{aligned} \tag{20}$$

Demanders consume the vector (.43750, .073225) and receive utility .505363, while suppliers consume the vector (.56250, .323225) and receive utility .78125. There are a total of four consumers of each type over the two rounds, so the market maker consumes the vector (0, .4142); because there is no aggregate uncertainty, the market maker sells all the first-period consumption he acquires in all states of nature. Note that neither prices nor utilities depend on the parameter γ or on whether we are in event α or β.

Under the Walrasian auctioneer, the prices that prevail during a particular round depend on whether α or β occurs, and they depend on the parameter γ. When α occurs we have $p_t = (1 - \gamma)/2$ and when β occurs we have $p_t = (1 + \gamma)/2$.

Consumption and utility under the Walrasian-auctioneer system are as follows. When α occurs, demanders consume the vector

$$([3 + \gamma^2]/8, [1 + \gamma]/4)$$

and receive utility

$$(\gamma^2 + 2\gamma + 9)/16.$$

Suppliers consume the vector

$$([5 - 2\gamma + \gamma^2]/8, [1 + \gamma]/4)$$

and receive utility

$$(\gamma^2 - 2\gamma + 13)/16.$$

It is easy to verify that the market clears.

When β occurs, demanders consume the vector

$$([3 + \gamma^2]/8, [1 - \gamma]/4)$$

and receive utility

$$(\gamma^2 - 2\gamma + 9)/16.$$

Suppliers consume the vector

$$([5 + 2\gamma + \gamma^2]/8, [1 - \gamma]/4)$$

and receive utility

$$(\gamma^2 + 2\gamma + 13)/16.$$

Once again, the market clears.

It might appear that consumers actually prefer price volatility (high γ), because straight averages of utility under α and utility under β yield functions that are strictly increasing in γ. This appearance is deceiving, because demanders are more likely to find themselves in event β, where demanders are plentiful, than in event α, where demanders are scarce. Assuming that all demanders are identical (ex ante) and face the same probabilities of arriving in a particular event, the probability of a demander being in event α is $(1 - \gamma)/2$ and the probability of a demander being in event β is $(1 + \gamma)/2$. The probability of a supplier being in event α is $(1 + \gamma)/2$ and the probability of a supplier being in event β is $(1 - \gamma)/2$.

The correct calculation yields an expected utility for the demanders of $[(\gamma^2 + 2\gamma + 9)/16](1 - \gamma)/2 + [(\gamma^2 - 2\gamma + 9)/16](1 + \gamma)/2$, which simplifies to

$$(9 - \gamma^2)/16.$$

A similar calculation yields an expected utility for the suppliers of

$$(13 - \gamma^2)/16.$$

For both consumers, expected utility decreases with the imbalance parameter γ. In the limit, when $\gamma = 1$, there is no trade and the above expressions for expected utility reduce to the autarky utility level.

We are now ready to compare the welfare properties of the market-maker system and the Walrasian-auctioneer system. For values of γ greater than .707, suppliers prefer the market-maker system; for values of γ greater than .956, both suppliers and demanders prefer the market-maker system.

Example 2 establishes the possibility of a monopolistic market maker leading to an allocation that Pareto dominates (in terms of ex ante expected utility) the allocation resulting from a Walrasian auctioneer. The advantages of price smoothing offered by a more liquid market can outweigh the surplus extracted by the monopolist. Indeed, the cutoff value for γ would be even lower if we introduced more risk aversion into Example 2. Taking the current utility function, in which the marginal utility of first-period consumption is constant, and applying a monotonic, concave transformation leads to the same observable behavior but lowers the cutoff at which the monopolist is preferred. By making the consumers sufficiently risk averse, the cutoff can be reduced to .293 for suppliers and .707 for demanders.

5. THE MARKET MAKER VERSUS AN AUCTION MARKET WITH A SOPHISTICATED TRADER

The comparison between the market maker and the auctioneer in Section 4 was uneven, because the market maker had better information about supply and demand shocks than anyone at the auction. To compare the market maker to the auctioneer on a more even footing, we allow anyone to learn the stochastic process for supply and demand and to maintain a continuous market presence. The idea is to isolate the extent to which the market maker can be preferred to the auctioneer due to institutional differences rather than exogenously imposed informational advantages.

Here we analyze an auction market where one of the traders is "sophisticated" like our market maker. Learning the stochastic process for supply and demand and maintaining a market presence requires the payment of a fixed cost, C. Without uncertainty, there is no distinction among a single sophisticated trader choosing a demand schedule, choosing a quantity, or choosing a price. When the revelation of information is involved, the precise specification of the traders' actions will be important. We consider a sophisticated player who strategically participates in each auction, in effect choosing a demand schedule.

In what follows, Example 3 extends Example 2 to the case of an auction market with a sophisticated trader. On the basis of these examples, we consider the outcome of competition between the Walrasian system (where a trader chooses whether or not to become sophisticated) and the market-maker system. Based on the underlying parameters, we can make a preliminary evaluation of which system is likely to prevail: a market-maker system, an auction system without a sophisticated trader, or an auction system with a sophisticated trader.

EXAMPLE 3. This example has consumers with the same endowments and linear-quadratic preferences as those in Examples 1 and 2. The stochastic processes for supply and demand are as in Example 2. In particular, there is no aggregate uncertainty, with events α and β each known to occur once. A sophisticated trader without initial endowments can participate in the two auctions.

Allowing the sophisticated trader to announce his desired trade at each price is equivalent to allowing him to infer whether the event is α or β and to select the equilibrium price, with his own trade being the market clearing quantity. For example, when α occurs demand and supply are given by Eq. (14). The sophisticated trader's excess demand equals the rest of the market's excess supply, which equals $(1 + \gamma)p_\alpha - (1 - \gamma)(1 - p_\alpha)$. This expression can be simplified to $(2p_\alpha - 1 + \gamma)$. We have the following maximization problem, where p_α now refers to the price in event α and p_β now refers to the price in event β:

$$\max(1 - p_\alpha)(2p_\alpha - 1 + \gamma) + (1 - p_\beta)(2p_\beta - 1 - \gamma) \tag{21}$$

subject to

$$p_\alpha(2p_\alpha - 1 + \gamma) + p_\beta(2p_\beta - 1 - \gamma) \leq 0. \tag{22}$$

When $\gamma = .5$, the optimal solution is $p_\alpha = .4045$ and $p_\beta = .6245$. Expected utilities are

demanders: .5445, suppliers: .8075, soph. player: .1180. (23)

We now consider the likely result of competition between a market-maker system and a Walrasian-auction system with and without a sophisticated trader. We use the following procedure. Given an auction system, someone will invest to become sophisticated if and only if profits exceed C. This determines whether or not our "candidate" auction system will involve a sophisticated trader. If the market maker's profits do not exceed C, then the candidate auction system will prevail.

If the market maker's profits exceed C, then the market-maker system is also a candidate. When suppliers and demanders prefer the market-maker system to the candidate auction system, the market maker prevails; when suppliers and demanders prefer the auction system, then the auction system prevails. When the two systems are Pareto incomparable, then the outcome is ambiguous.

Two facts emerge from a comparison of utilities and profits in Examples 2 and 3. First, the market maker's profits are greater than the sophisticated trader's profits. This fact is quite robust and certainly holds whenever there is no aggregate uncertainty. Second, suppliers and demanders prefer the sophisticated trader to the market maker. For Examples 2 and 3, the price prevailing in the market with the sophisticated trader is always strictly within the spread offered by the market maker. Therefore, we have the following:

(1) Whenever C is less than the profits of the sophisticated trader, then it pays to become sophisticated. A market maker will not survive, be-

cause someone will pay the cost and join an auction market, offering consumers higher utility.

(2) When C exceeds the market maker's profits, the auction market without a sophisticated trader prevails.

(3) When C is between the sophisticated trader's profits and the market maker's profits, and when imbalances between supply and demand within a round are large, then someone will invest and become a market maker. The market maker survives because no one will pay to become a sophisticated trader, and the market maker provides higher expected utility than the auctioneer.

(4) When C is between the sophisticated trader's profits and the market maker's profits, and when imbalances within a round are small, the auctioneer prevails.

(5) The outcome is ambiguous when C is between the sophisticated trader's profits and the market maker's profits, and when imbalances within a round are neither small nor large.

6. CONCLUDING REMARKS

The multi-round model captures the fact that a market maker receives information and makes many decisions within the market period. It recognizes the possibility of temporary imbalances between supply and demand arising within the market period. The results indicate that the advantages to the consumers of having a market maker instead of round-by-round market clearing are greatest when imbalances between supply and demand within a round are likely, when overall supply and demand can be predicted with some certainty, and when the cost of acquiring information is moderately high. When these conditions are met, a market maker provides continuity by maintaining a constant price in the face of shifting supply and demand, and he provides depth by allowing a big player to make a large trade all in one round without adversely affecting the price.

It should be noted that the market maker modeled in this paper is not a "specialist" per se. There is probably no real world market whose rules can be described by a model this simple. The purpose of this paper is to model liquidity in a general equilibrium setting, showing the advantages of an intermediary over mechanical, round-by-round market clearing.

Understanding the precise institutions of actual markets with this approach still remains a distant goal. However, this goal should be strived for. Domowitz (1989) describes the algorithms of three automated trade execution systems. It would be interesting to analyze one of these automated systems according to the present approach, by specifying the arrival process of supply and demand within trading rounds and modeling the decision of whether or not to become sophisticated. Such an analysis would be complicated by the complexity of these algorithms and the presence of several strategic players.

There is every reason to believe that the advantages of an intermediary over a Walrasian auctioneer in providing liquidity would be enhanced when there is asymmetric information across consumers. Diamond and Dybvig (1983) consider a model of banking where a bank can improve on the market by offering insurance that the market cannot provide, due to adverse selection. Gorton and Pennachi (1990) examine a model where some consumers face an immediate consumption requirement. Some investors have inside information about the return of a risky asset. An intermediary improves upon the provision of liquidity by holding the risky asset and splitting the cash flow into a safer stream and a riskier stream.

There are two aspects of liquidity that this paper attempts to separate. Money is liquid because it is accepted as a medium of exchange. Other papers, such as Kiyotaki and Wright (1989), explain why money is accepted as a medium of exchange. Here, we take the demand for an immediate transaction as given and show how some market structures can provide greater liquidity than others. It is useful to think of the numeraire commodity as money, but any commodity for which an immediacy demand exists will do. The liquidity of markets is a meaningful concept even if money is not one of the commodities.

APPENDIX

Proof of Proposition 2.1. For convenience we adopt the following notation

$$z_1(p^{\text{bid}}) \equiv \frac{(1 - p^{\text{bid}})S'(p^{\text{bid}}) - S(p^{\text{bid}})}{p^{\text{bid}}S'(p^{\text{bid}}) + S(p^{\text{bid}})}$$

$$z_2(p^{\text{ask}}) \equiv \frac{(1 - p^{\text{ask}})D'(p^{\text{ask}}) - D(p^{\text{ask}})}{p^{\text{ask}}D'(p^{\text{ask}}) + D(p^{\text{ask}})}\,.$$

Y_1 is the factor by which $\theta_1, \theta_2, \ldots, \theta_m$ are increased, and Y_2 is the factor by which $\phi_1, \phi_2, \ldots, \phi_n$ are increased.

The interior first-order conditions can now be rewritten as

$$H_1(p^{\text{bid}}, p^{\text{ask}}; I, Y_1, Y_2) \equiv [z_1(p^{\text{bid}})(\textstyle\sum_{i=1}^{m} q_i\theta_i)/\theta^m] - [z_2(p^{\text{ask}})(\textstyle\sum_{j=1}^{n} r_j\phi_j)/\phi_1] = 0 \qquad \text{(A.1)}$$

and

$$H_2(p^{\text{bid}}, p^{\text{ask}}; I, Y_1, Y_2) \equiv p^{\text{bid}} Y_1 \theta_m S(p^{\text{bid}}) - p^{\text{ask}} Y_2 \phi_1 D(p^{\text{ask}}) - I = 0. \tag{A.2}$$

CLAIM 1.

$$dz_1(p^{\text{bid}})/dp^{\text{bid}} < 0 \qquad \text{and} \qquad dz_2(p^{\text{ask}})/dp^{\text{ask}} < 0.$$

Proof of Claim 1 (suppressing the functional dependencies where obvious).

$$\frac{dz_1(p^{\text{bid}})}{dp^{\text{bid}}}$$
$$= \frac{(p^{\text{bid}}S' + S)[(1 - p^{\text{bid}})S'' - 2S'] - [(1 - p^{\text{bid}})S' - S](p^{\text{bid}}S'' + 2S')}{(p^{\text{bid}}S' + S)^2}.$$

This expression simplifies to

$$dz_1(p^{\text{bid}})/dp^{\text{bid}} = [SS'' - 2(S')^2]/(p^{\text{bid}}S' + S)^2. \tag{A.3}$$

Since we know $S'' \leq 0$ holds, Eq. (A.3) implies $dz_1(p^{\text{bid}})/dp^{\text{bid}} < 0$. A similar equation yields

$$dz_2(p^{\text{ask}})/dp^{\text{ask}} = [DD'' - 2(D')^2]/(p^{\text{ask}}D' + D)^2. \tag{A.4}$$

The numerator of the right side of Eq. (A.4) must be negative at any interior solution. This is because payments of second-period consumption to demanders are strictly decreasing in p^{ask}, so the market maker will never decrease p^{ask} to the point at which less first-period consumption is purchased. Therefore, we have $d[p^{\text{ask}}D(p^{\text{ask}})]/dp^{\text{ask}} < 0$, which implies

$$p^{\text{ask}}D' + D < 0. \tag{A.5}$$

From our maintained assumptions on demand, it follows that

$$p^{\text{ask}}D'' + 2D' < 0 \text{ holds.} \tag{A.6}$$

Inequalities (A.5) and (A.6) together imply $DD'' - 2(D')^2 < 0$, which together with (A.4) guarantee $dz_2(p^{\text{ask}})/dp^{\text{ask}} < 0$, completing the proof of Claim 1.

The Jacobian matrix

$$J = \begin{pmatrix} \partial H_1/\partial p^{\text{bid}} & \partial H_1/\partial p^{\text{ask}} \\ \partial H_2/\partial p^{\text{bid}} & \partial H_2/\partial p^{\text{ask}} \end{pmatrix}$$

equals

$$\begin{bmatrix} \dfrac{\sum_{i=1}^{m} q_i\theta_i}{\theta_m} \dfrac{dz_1(p^{\text{bid}})}{dp^{\text{bid}}} & \dfrac{-\sum_{j=1}^{n} r_j\phi_j}{\phi_1} \dfrac{dz_2(p^{\text{ask}})}{dp^{\text{ask}}} \\ \theta_m[p^{\text{bid}}S' + S]Y_1 & -\phi_1[p^{\text{ask}}D' + D]Y_2 \end{bmatrix}.$$

From Claim 1, the terms in J have the signs

$$\begin{pmatrix} - & + \\ + & + \end{pmatrix}$$

so $|J| < 0$.

We can now apply the implicit function theorem to conclude:

$$\partial p^{\text{ask}}/\partial I = -1/|J| \begin{vmatrix} \dfrac{\sum_{i=1}^{m} q_i\theta_i}{\theta_m} \dfrac{dz_1(p^{\text{bid}})}{dp^{\text{bid}}} & 0 \\ \theta_m[p^{\text{bid}}S' + S]Y_1 & -1 \end{vmatrix} > 0$$

$$\partial p^{\text{bid}}/\partial I = -1/|J| \begin{vmatrix} 0 & \dfrac{-\sum_{j=1}^{n} r_j\phi_j}{\phi_1} \dfrac{dz_2(p^{\text{ask}})}{dp^{\text{ask}}} \\ -1 & -\phi_1[p^{\text{ask}}D' + D]Y_2 \end{vmatrix} > 0$$

$$\partial p^{\text{bid}}/\partial Y_1 = -1/|J| \begin{vmatrix} 0 & \dfrac{-\sum_{j=1}^{n} r_j\phi_j}{\phi_1} \dfrac{dz_2(p^{\text{ask}})}{dp^{\text{ask}}} \\ p^{\text{bid}}\theta_m S(p^{\text{bid}}) & -\phi_1[p^{\text{ask}}D' + D]Y_2 \end{vmatrix} < 0$$

$$\partial p^{\text{ask}}/\partial Y_1 = -1/|J| \begin{vmatrix} \dfrac{\sum_{i=1}^{m} q_i\theta_i}{\theta_m} \dfrac{dz_1(p^{\text{bid}})}{dp^{\text{bid}}} & 0 \\ \theta_m[p^{\text{bid}}S' + S]Y_1 & p^{\text{bid}}\theta_m S(p^{\text{bid}}) \end{vmatrix} < 0$$

$$\partial p^{\text{bid}}/\partial Y_2 = -1/|J| \begin{vmatrix} 0 & \dfrac{-\sum_{j=1}^{n} r_j\phi_j}{\phi_1} \dfrac{dz_2(p^{\text{ask}})}{dp^{\text{ask}}} \\ -p^{\text{ask}}\phi_1 D(p^{\text{ask}}) & -\phi_1[p^{\text{ask}}D' + D]Y_2 \end{vmatrix} > 0$$

$$\partial p^{\text{ask}}/\partial Y_2 = -1/|J| \begin{vmatrix} \dfrac{\sum_{i=1}^{m} q_i\theta_i}{\theta_m} \dfrac{dz_1(p^{\text{bid}})}{dp^{\text{bid}}} & 0 \\ \theta_m[p^{\text{bid}}S' + S]Y_1 & -p^{\text{ask}}\phi_1 D(p^{\text{ask}}) \end{vmatrix} > 0.$$

We have demonstrated parts (a)–(c) of Proposition 2.1.

Proof of Proposition 3.1. Consider the problem in which all of the shocks are known in advance.

$$\max \textstyle\sum_{t=1}^{T}(1 - p_t^{\text{bid}})S(p_t^{\text{bid}})\theta^t - \sum_{t=1}^{T}(1 - p_t^{\text{ask}})D(p_t^{\text{ask}})\phi^t \tag{A.7}$$

subject to

$$0 \leq p_t^{\text{bid}} \leq 1 \qquad \text{and} \qquad 0 \leq p_t^{\text{ask}} \leq 1 \tag{A.8}$$

$$\textstyle\sum_{t=1}^{T} p_t^{\text{bid}}\theta^t S(p_t^{\text{bid}}) \leq I + \sum_{t=1}^{T} p_t^{\text{ask}}\phi^t D(p_t^{\text{ask}}). \tag{A.9}$$

If $I \geq (\sum_{t=1}^{T} \theta^t)\bar{p}^{\text{bid}}S(\bar{p}^{\text{bid}})$ occurs we are at a corner solution where $p_t^{\text{ask}} = 1$ and $p_t^{\text{bid}} = \bar{p}^{\text{bid}}$ for all t ($\bar{p}^{\text{bid}}$ is the unconstrained profit maximizing price discussed earlier). Prices are constant and equal to the solution of the one-round model with supply $(\sum_{t=1}^{T} \theta^t)S(\bar{p}^{\text{bid}})$ and demand $(\sum_{t=1}^{T} \phi^t)D(p^{\text{ask}})$.

If $I < (\sum_{t=1}^{T} \theta^t)\bar{p}^{\text{bid}}S(\bar{p}^{\text{bid}})$ occurs we are at a strictly interior solution. The first-order conditions are given by Eq. (A.9) and

$$\lambda = \frac{(1 - p_t^{\text{bid}})S'(p_t^{\text{bid}}) - S(p_t^{\text{bid}})}{p_t^{\text{bid}}S'(p_t^{\text{bid}}) + S(p_t^{\text{bid}})} = \frac{(1 - p_t^{\text{ask}})D'(p_t^{\text{ask}}) - D(p_t^{\text{ask}})}{p_t^{\text{ask}}D'(p_t^{\text{ask}}) + D(p_t^{\text{ask}})} \tag{A.10}$$

for all t, where λ is the Lagrangean multiplier on (A.9). From Claim 1 of the proof of Proposition 2.1, we know that the expressions on the right side of (A.10) are strictly decreasing in p_t^{bid} and p_t^{ask}, respectively. Since these expressions equal λ for all t, it follows that $p_t^{\text{bid}} = p_{t'}^{\text{bid}}$ and $p_t^{\text{ask}} = p_{t'}^{\text{ask}}$ for all t and t'. Thus, prices are constant. Since the first-order conditions (at constant prices) are identical to those of the one-round model with supply $(\sum_{t=1}^{T} \theta^t)S(p^{\text{bid}})$ and demand $(\sum_{t=1}^{T} \phi^t)D(p^{\text{ask}})$, we must have the same prices.

We have shown that the conclusion of Proposition 3.1 is true when *all the shocks are known*. We now show the same is true whenever $\sum_{t=1}^{T} \theta^t$ and $\sum_{t=1}^{T} \phi^t$ are known. Let $p_t^{\text{bid}} = \overset{*}{p}^{\text{bid}}$ and $p_t^{\text{ask}} = \overset{*}{p}^{\text{ask}}$ be the solution to problems (A.7)–(A.9), in which the shocks are known.

Since prices are constant, they can be factored out of the sums in (A.7) and (A.9). That is, $\overset{*}{p}^{\text{bid}}$ and $\overset{*}{p}^{\text{ask}}$ depend on the state of nature only through the sums $\sum_{t=1}^{T} \theta^t$ and $\sum_{t=1}^{T} \phi^t$. It is therefore feasible to choose the same prices in (7)–(9) with limited information as the market maker would choose under complete information. Since complete information expands the set of feasible choices, the optimal solution must be at least as high as the optimal solution with limited information. It follows that $\overset{*}{p}^{\text{bid}}$ and $\overset{*}{p}^{\text{ask}}$ are optimal choices when $\sum_{t=1}^{T} \theta^t$ and $\sum_{t=1}^{T} \phi^t$ are known.

Proof of Proposition 3.2. It is enough to show that, at the optimal prices for (7)–(9) $\overset{*}{p}^{\text{bid}}$ and $\overset{*}{p}^{\text{ask}}$, constraint (10) is satisfied for $\tau = 1, 2, \ldots, T-1$. We know that (10) is satisfied for $\tau = 0$, which implies

$$\overset{*}{p}^{\text{bid}} S(\overset{*}{p}^{\text{bid}}) \leq I/\textstyle\sum_{t=1}^{T} \theta^t + \overset{*}{p}^{\text{ask}} D(\overset{*}{p}^{\text{ask}})(\sum_{t=1}^{T} \phi^t)/(\sum_{t=1}^{T} \theta^t). \quad \text{(A.11)}$$

Since a nonnegative inventory constraint presumes initial inventory is nonnegative, (A.11) implies, for all τ,

$$\overset{*}{p}^{\text{bid}} S(\overset{*}{p}^{\text{bid}}) \leq I/\textstyle\sum_{t=1}^{T-\tau} \theta^t + \overset{*}{p}^{\text{ask}} D(\overset{*}{p}^{\text{ask}})(\sum_{t=1}^{T} \phi^t)/(\sum_{t=1}^{T} \theta^t). \quad \text{(A.12)}$$

From (11) and (A.12), we have, for all τ,

$$\overset{*}{p}^{\text{bid}} S(\overset{*}{p}^{\text{bid}}) \leq I/\textstyle\sum_{t=1}^{T-\tau} \theta^t + \overset{*}{p}^{\text{ask}} D(\overset{*}{p}^{\text{ask}})(\sum_{t=1}^{T-\tau} \phi^t)/(\sum_{t=1}^{T-\tau} \theta^t). \quad \text{(A.13)}$$

Since (A.13) is satisfied for all τ, constraint (10) also holds for all τ.

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